Integral approach to sensitive singular perturbations

Nicolas Meunier, Evariste Sanchez-Palencia

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Nicolas Meunier* and E. Sanchez-Palencia†

Abstract

We consider singular perturbation elliptic problems depending on a parameter ε such that, for $\varepsilon = 0$ the boundary conditions are not adapted to the equation (they do not satisfy the Shapiro - Lopatinskii condition). The limit only holds in very abstract spaces out of distribution theory involving complexification and non-local phenomena. We give a very elementary model problem showing the main features of the limit process, as well as a heuristic integral procedure for obtaining a description of the solutions for small ε. Such kind of problems appear in thin shell theory when the middle surface is elliptic and the shell is fixed by a part of the boundary and free by the rest.

1 Introduction

The main purpose of this paper is to give general ideas on a kind of singular perturbations arising in thin shell theory when the middle surface is elliptic and the shell is fixed by a part of the boundary and free by the rest as well as an integral heuristic procedure reducing them to simpler problems. The system depends drastically on the parameter ε equal to the relative thickness of the shell. It appears that the ”limit problem” for $\varepsilon = 0$ is highly ill-posed. Indeed, the boundary conditions on the free boundary are not ”adapted” to the system of equations; they do not satisfy the Shapiro - Lopatinskii (SL hereafter) condition. Roughly speaking, this amounts to some kind of ”transparency” of the boundary conditions, which allow some kind of locally indeterminate oscillations along the boundary, exponentially decreasing inside the domain. This pathological behavior is only concerned with $\varepsilon = 0$. In fact, for $\varepsilon > 0$ the problem is ”classical”. When ε is positive but small, the ”determinacy” of the oscillations only holds with the help of boundary conditions on other boundaries, as well as the small terms coming from $\varepsilon > 0$.

In such kind of situations, the limit problem has no solution within classical theory of partial differential equations, which is uses distribution theory. It is sometimes possible to prove the convergence of the solutions u^ε towards some limit u^0, but this ”limit solution” and the topology of the convergence are concerned with abstract spaces not included in the distribution space.

After recalling the SL condition (section 2), we give in section 3 a very simple example of such a perturbation problem. The geometry of the domain (an infinite strip) allows explicit treatment by Fourier transform in the longitudinal direction. The inverse Fourier transform within distribution theory is only possible for $\varepsilon > 0$, whereas for $\varepsilon = 0$ it is only possible in the framework of analytic functionals (higly singular and not enjoying localization properties). This example shows the prominent role of components with high frequency; for small ε, the ”smooth parts” (i. e. with small $|\xi|$) of

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the solutions may be neglected with respect to ”singular ones” (i. e. with large $|\xi|$). We also recall an example of elliptic Cauchy problem (in fact Hadamard’s counter-example) which exhibits some relation with the limit problem.

In section 4, we report the heuristic procedure of [EgMeSa07]. In this latter article, we addressed a more complicated problem including a variational structure, somewhat analogous to the shell problem, but simpler, as concerning an equation instead of a system. It is shown that the limit problem contains in particular an elliptic Cauchy problem. This problem was handled in both a rigorous (very abstract) framework and using a heuristic procedure for exhibiting the structure of the solutions with very small ε. The reasons why the solution goes out of the distibution space as ε goes to 0 are then evident. In section 4 we present a simplified version of the heuristic procedure involving only the essential facts of the approximation, which are very much analogous to the method of construction of a parametrix in elliptic problems [Ta81], [EgSc97]:

-Only principal (with higher differentiation order) terms are taken into account.

-Locally, the coefficients are considered to be constant, their values being frozen at the corresponding points.

-After Fourier transform ($x \to \xi$), terms with small ξ are neglected with respect to those with larger ξ (which amounts to taking into account singular parts of the solutions while neglecting smoother ones). We note that this approximation, aside with the two previous ones, lead to some kind of ”local Fourier transform” which we shall use freely in the sequel.

Another important ingredient of the heuristics is a previous drastic restriction of the space where the variational problem is handled. In order to search for the minimum of energy, we only take into account functions such that the energy of the limit problem is very small. This is done using a boundary layer method within the previous approximations, i. e. for large $|\xi|$. This leads to an approximate simpler formulation of the problem for small ε, where it is apparent that the limit problem involves a smoothing operator and cannot have a solution within distribution theory.

It should prove useful to give an example of a sequence of functions converging to an analytical functional (but going out of the distribution space, then leading to a ”complexification” phenomenon). It is known ([Sc50], [GeCh64]) that (direct and inverse) Fourier transform within distribution theory is only possible for temperate distributions, not allowing functions with exponential growth at infinity. The space of (direct or inverse) Fourier transform of general distributions is noted Z'. It is a space of analytical functionals: the corresponding test functions are analytical rapidly decreasing functions, forming the space Z.

Let us consider the (non temperate) distribution (or function) $\hat{u}(\xi) = \cosh(\xi)$. The sequence

$$\hat{u}^\lambda(\xi) = \begin{cases} \cosh(\xi) \text{ if } |\xi| < \lambda, \\ 0 \text{ otherwise} \end{cases}$$

converges to $\hat{u}$ in the distribution sense as λ goes to infinity. The inverse Fourier transforms $u^\lambda(x)$ converge in Z' to the analytical functionnal $u(x)$. The functions $\hat{u}^\lambda(\xi)$ are tempered and their inverse Fourier transforms are easily computed by hand. It appears that for large λ

$$u^\lambda(x) \approx \frac{e^\lambda}{2\pi} \frac{1}{1+x^2} (\cos(\lambda x) + x \sin(\lambda x)).$$

It is then apparent that $u^\lambda(x)$ consists of a ”nearly periodic” function with period tending to zero along with $1/\lambda$, multiplied by an ”envelop” defined by $\frac{1}{1+x^2}$ and by the factor $\frac{e^\lambda}{2\pi}$. Moreover, it should be noted that the amplitude is exponentially large with respect to the inverse of the period. It

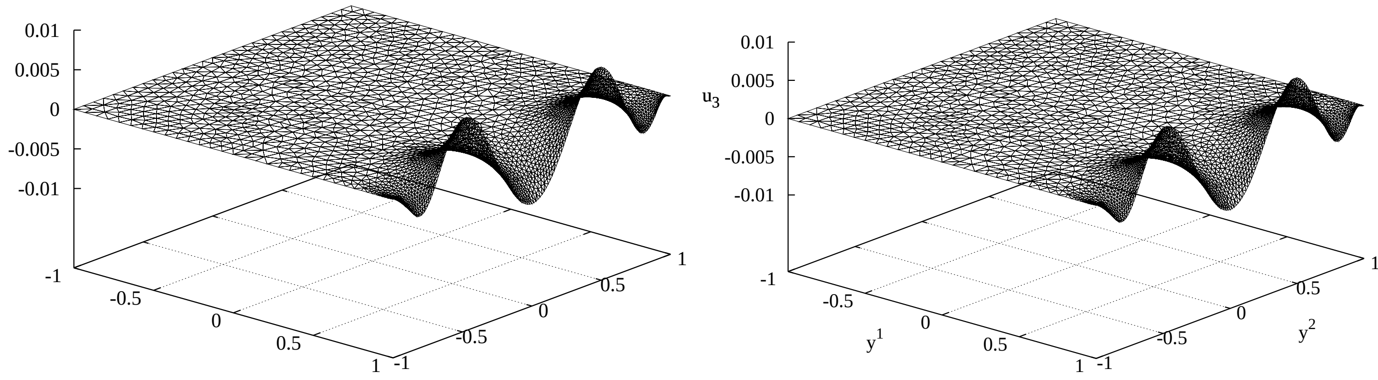

Figure 1.1: Normal displacement for $\varepsilon = 10^{-3}$ on the left and for $\varepsilon = 10^{-5}$ on the right

is then apparent that the limit is an "extremely singular" function as the "graph" fills the entire plane. Moreover, it is clear (and may be rigorously proved [EgMeSa07] that the sequence u^{λ} goes out of the distribution space everywhere, not only in the vicinity of $x = 0$ as is suggested by the formal inverse Fourier transform of $\cosh(\xi) = \Sigma_{n=0}^{+\infty} \frac{\xi^{2n}}{(2n)!}$, which is

$$u(x) = \Sigma_{n=0}^{+\infty} \frac{-i}{(2n)!} \delta^{2n}(x),$$

apparently a singularity "of order infinity" at the origin. This fact constitutes an example of the property that elements of Z' can only be tested with analytical test functions, then not enjoying localization properties.

The motivation for studying that kind of problems comes from shell theory, see [SaHuSa97], [BeMiSa08]. It appears that when the middle surface is elliptic (both principal curvatures have same sign) and is fixed by a part Γ_0 of the boundary and free by the rest Γ_1, the "limit problem" as the thickness ε tends to zero is elliptic, with boundary conditions satisfying SL on Γ_0, and boundary not satisfying SL on Γ_1. Without going into details, which may be found in [MeSa06], [MeSaHuSa07], [EgMeSa07] and [EgMeSa09], we show numerical computations taken from [BeMiSa08] of the normal displacement for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-5}$ (figures 1.1 on the left and on the right respectively) when the shell is acted upon by a normal density of forces on a rectangular region of the plane of parameters. The most important feature is constituted by large oscillations nearby the free boundary Γ_1. It is apparent that, when passing from $\varepsilon = 10^{-3}$ to $\varepsilon = 10^{-5}$, the amplitude of the oscillations grows from 0.001 to 0.01. The singularities produced by the jump of the applied forces inside the domain is still apparent for $\varepsilon = 10^{-3}$, not for $\varepsilon = 10^{-5}$, where only oscillations along the boundary are visible. Moreover, the number of such oscillations pass from nearly 3 for $\varepsilon = 10^{-3}$ to nearly 5 for $\varepsilon = 10^{-5}$ and is then nearly proportional to $\log(1/\varepsilon)$. We shall see that all these features agree with our theory.

2 The Shapiro - Lopatinskii condition for boundary conditions of elliptic equations

In this section, we recall some properties of elliptic PDE, see [AgDoNi59] and [EgSc97] for more details.

We consider a PDE of the form

$$P(x,\partial_\alpha)u = f(x) \tag{2.1}$$

Where $x = (x_1, x_2)$ and $\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$, and P is a polynomial of degree $2m$ in ∂_α. Let P_0 be the ”principal part”, i. e., the terms of higher order. The equation is said to be elliptic at x if the homogeneous polynomial of degree $2m$ in ξ_α:

$$P_0(x, -i\xi_\alpha) = 0 \tag{2.2}$$

has no solution $\xi = (\xi_1, \xi_2) \neq (0, 0)$ with real ξ_α. When the coefficients are real (this is the only case that we shall consider) this implies that the degree is even (this is the reason why we denoted it by $2m$). The left hand side of (2.2) is said to be the ”principal symbol”; the ”symbol” is obtained in an analogous way taking the whole P instead of the principal part P_0. We note that replacing $\partial/\partial x_\alpha$ by $-i\xi_\alpha$ in P_0 amounts to taking formally the Fourier transform $x \to \xi$ for the homogeneous equation with constant coefficients obtained by discarding the lower order terms and freezing the coefficients at x. Obviously, ellipticity on a domain Ω is defined as elliptic at any $x \in \Omega$.

It is worthwhile mentioning that ellipticity amounts to non - existence of ”travelling waves” of the form

$$e^{-i\xi x} \tag{2.3}$$

for the equation obtained after discarding lower order terms and freezing coefficients. Here ”travelling” amounts to ”with real ξ”; note that solutions as (2.3) with non real ξ are necessarily exponentially growing or decaying (in modulus) in some direction. Moreover, when a solution of the form (2.3) exists (with ξ either real or not), it also exists for $c\xi$ with any c. In a heuristic framework,we may suppose that $|\xi|$ is very large; this justify to discard lower order terms (= of lower degree in $|\xi|$). In the same (heuristic) order of ideas, freezing the coefficients allows to consider ”local solutions”. This amounts to multiply the solutions by a ”cutoff” function $\theta(x)$ or equivalently taking the convolution of the Fourier transform with $\hat{\vartheta}(\xi)$, which do not modify the behavior for large ξ. Microlocal analysis gives a rigorous sense to that heuristics. It then appears that local singularities of a solution u (associated with behavior of the Fourier transform for large $|\xi|$) cannot occur in elliptic equations unless they are controlled by the (Fourier transform of the) right hand side f. This gives a ”heuristic proof” of the classical property that local solutions of elliptic equations are rigorously associated with singularities of f.

What happens with solutions near the boundary? Local Fourier transform is no longer possible, but, after rectification of the boundary in the neighborhood of a point, we may perform a tangential Fourier transform. If, for instance, the considered part of the boundary is on the axis x_1 and the domain is on the side $x_2 > 0$, taking only higher order terms and frozen coefficients, we have solutions of the form (2.3) with real ξ_1 (as coming from the Fourier transform) and non - real ξ_2. The dependence in x_2 is immediately obtained by solving an ODE with constant coefficients. Obviously, the solutions are exponentially growing or decreasing for $x_2 > 0$. As the coefficients are real, there are precisely m (linearly independent) growing and m decreasing (in the case of multiple roots, dependence in x_2 of the form $x_2 e^{\lambda) x_2}$ and analogous also occur). Roughly speaking, there are solutions of the form:

$$\sum_k C_k e^{-i\xi_1 x_1} e^{\lambda_k x_2} \tag{2.4}$$

with real ξ_1 and $Re(\lambda) \neq 0$ (here k is running from 1 to $2m$) . Boundary conditions on $x_2 = 0$ should control solutions with $Re(\lambda) < 0$, i. e., exponentially decreasing inside the domain, whereas exponentially growing ones should be controlled ”by the equation in the rest of the domain and the boundary conditions on the other parts of the boundary". In other words, "good boundary conditions" should determine, (within our approximation of the half plane and frozen coefficients) the solutions of the equation of the form (2.4) with $Re(\lambda) < 0$. Obviously, the number of such boundary conditions is m. A set of m boundary conditions enjoying the above property is said to satisfy the Shapiro - Lopatinskii condition. There are several equivalent specific definitions of it. We shall mainly use the following one:

Definition 2.1. *Let P be elliptic at a point O of the boundary. A set of m boundary conditions $B_j(x, \partial_\alpha) = g_j(x)$, $j = 1, ...m$ is said to satisfy the SL condition at O when, after a local change to new coordinates with origin at O and axis x_1 tangent to the boundary, taking only the higher order terms and coefficients frozen at O in the equation and the boundary conditions, the solutions of the form (2.4) with $Re(\lambda) < 0$ obtained by formal tangential Fourier transform are well defined by the boundary conditions.*

Remark 2.1. *The above definition should be understood in the sense of formal solution for any given (real and non-zero) ξ_1. The SL condition is not concerned with solutions in certain spaces. It is purely algebraic, and concerns m conditions imposed to the m (decreasing with x_2) linearly independent solutions of the ODE obtained from P_0 by formal tangential Fourier transform. This also amounts to saying that imposing the boundary conditions equal to zero, the considered solutions must vanish. In fact, the SL condition amounts to non-vanishing of a certain determinant, and as so it is generically satisfied: conditions not satisfying it are rarely encountered. In particular, in "well-behaved problems", when coerciveness on appropriate spaces is proved, the SL condition is not usually checked. It should also be noted that the SL condition is independent of a change of variables, and, in most cases, the change is trivial. On the other hand, there are also definitions of the SL condition without change of variables. Last, it should also be noted that the SL condition has nothing to do with lower order terms and the right hand side of the boundary conditions (as ellipticity is only concerned with the principal symbol); it is merely a condition of adequation of the principal part of the boundary operators to the principal part of the equation.*

Let us consider, as an exercise, examples for the laplacian.

$$P = -\partial_1^2 - \partial_2^2 \tag{2.5}$$

The principal symbol is $\xi_1^2 + \xi_2^2$ so that the equation is elliptic of order 2, then $m = 1$. "Good boundary conditions" are in number of 1.

Let us try the boundary condition (Dirichlet):

$$u = 0. \tag{2.6}$$

Taking any point of the boundary and (x_1, x_2) with origin at that point, tangent and normal to the boundary respectively, the equation is the same as in the initial variables, and formal tangential Fourier transform gives

$$(\xi_1^2 - \partial_2^2)\hat{u}(\xi_1, x_2) = 0 \tag{2.7}$$

and the solutions are

$$\hat{u}(\xi_1, x_2) = C_1(\xi_1)e^{|\xi_1|x_2} + C_2(\xi_1)e^{|\xi_1|x_2}. \tag{2.8}$$

Taking only the exponentially decreasing for $x_2 > 0$ we only have

$$\hat{u}(\xi_1, x_2) = C_1(\xi_1)e^{-|\xi_1|x_2}. \tag{2.9}$$

Now, imposing the "tangential Fourier transform" of (2.6):

$$\hat{u}(\xi_1, 0) = 0, \tag{2.10}$$

we see that it vanish identically. Then, the Dirichlet boundary condition satisfies the SL condition for the laplacian.

The case of the Neumann boundary condition for the laplacian

$$\frac{\partial u}{\partial n} = 0. \tag{2.11}$$

is analogous. (Note also that the Fourier condition $(\frac{\partial u}{\partial n}) + au = g$ is the same, as only the higher order terms are taken in consideration). Proceeding as before, we have, instead of (2.10):

$$\partial_2 \hat{u}(\xi_1, 0) = -|\xi_1| C_1(\xi_1) = 0, \tag{2.12}$$

which also gives $C_1(\xi_1) = 0$ and then $\hat{u} = 0$. Thus, (2.10) satisfies SL for (2.5).

Oppositely, the boundary condition:

$$(\partial_s - i\partial_n)u = 0, \tag{2.13}$$

where s and n denote the arc of the boundary and the normal, does not satisfy the SL condition for the laplacian. Indeed, taking the new local axes, s and n become x_1 and x_2, and after tangential Fourier transform:

$$(-i\xi_1 - i\hat{\partial_2)u})(\xi_1, 0) = 0, \tag{2.14}$$

which, applied to (2.9) becomes:

$$(-i\xi_1 + i|\xi_1|)C_1(\xi_1) = 0. \tag{2.15}$$

we then see that $C_1(\xi_1)$ vanishes for negative ξ_1, but is arbitrary for positive ξ_1. In fact , the boundary condition (2.13) is "transparent" for solutions of the form (2.9) with positive ξ_1.

Remark 2.2. *As it is apparent in the last example, when the SL condition is not satisfied, there is some kind of "local non-uniqueness", where "local" recalls that only higher order terms are taken in consideration, and the coefficients are frozen at the considered point of the boundary.*

The SL condition appears as some previous condition for solving elliptic problems. It is apparent that some pathology is involved at points of the boundary where it is not satisfied.

Let us mention, before closing this section, that the boundary conditions may be different on different parts of the boundary specially on different connected components of it (when there are points of junction of the various regions, usually singularities appear at that points).

3 An explicit perturbation problem where the SL condition is not satisfied on a part of the boundary of the limit problem

Let Ω be the strip $(-\infty, +\infty) \times (0, 1)$ of the (x, y) plan. We denote by Γ_0 and Γ_1 the boundaries $y = 0$ and $y = 1$ respectively. We then consider the boundary value problem depending on the parameter ε:

$$\begin{cases} \triangle u^\varepsilon = 0 \text{ on } \Omega \\ u^\varepsilon = 0 \text{ on } \Gamma_0 \\ \partial_x u + (i + \varepsilon^2)\partial_y u = \varphi \text{ on } \Gamma_1 \end{cases} \tag{3.16}$$

where φ is the data of the problem. It is a given function of x, that we shall suppose sufficiently smooth, tending to 0 at infinity. We shall solve it by $x \rightarrow \xi$ Fourier transform; it is easily seen that we also have automatically $u \rightarrow 0$ for $x \rightarrow \infty$, which may be added to the boundary conditions.

The boundary condition on Γ_0 is the Dirichlet one, which satisfies SL for the laplacian. Oppositely, the boundary condition on Γ_1 satisfies it for $\varepsilon > 0$ (this is easily checked), not at the limit $\varepsilon = 0$ (see the end of the previous section). The problem is to solve for $\varepsilon > 0$ and to study the behavior for ε going to zero.

Denoting by $\hat{}$ the $x \rightarrow \xi$ Fourier transform, $\hat{u}^\varepsilon$ is defined on the same Ω domain, but of the (ξ, y) plane. The solutions of the (transform of) equation and the boundary condition on Γ_0 are of the form

$$\hat{u}^\varepsilon(\xi, y) = \alpha(\xi) \sinh(\xi y) \tag{3.17}$$

where α denotes an unknown function to be determined with the boundary condition on Γ_1. It will prove useful to write the solution under the form

$$\hat{u}^\varepsilon(\xi, y) = \hat{\beta}^\varepsilon(\xi) \frac{\sinh(\xi y)}{\sinh(\xi)} \tag{3.18}$$

for the new unknown $\hat{\beta}^\varepsilon(\xi)$, which is the transform of the trace $u^\varepsilon(x, 0)$. Imposing the Fourier transform of the boundary condition on Γ_1 we have:

$$-i\xi \hat{\beta}^\varepsilon(\xi) + (i + \varepsilon^2) \frac{\cosh(\xi)}{\sinh(\xi)} \hat{\beta}^\varepsilon(\xi)\xi = \hat{\varphi}(\xi). \tag{3.19}$$

So that:

$$\hat{\beta}^\varepsilon(\xi) = \frac{\hat{\varphi}(\xi)}{-i\xi\Big(1 - \coth(\xi)\Big) + \varepsilon^2 \xi \coth(\xi)}. \tag{3.20}$$

In order to study this function, we should keep in mind that the expression $(1 - \coth(\xi))$ decays for $\xi \rightarrow +\infty$ as $2e^{-2\xi}$. Then, at the limit $\varepsilon = 0$ we have

$$\hat{\beta}^0(\xi) = \frac{\hat{\varphi}(\xi)}{-i\xi(1 - \coth(\xi))}. \tag{3.21}$$

For $\xi \rightarrow +\infty$ this function behaves as

$$\hat{\beta}^0(\xi) \approx 2 \frac{\hat{\varphi}(\xi)}{-i\xi} e^{2\xi}. \tag{3.22}$$

This shows (unless in the case of very special data φ with very fast decaying Fourier transform) that $\hat{\beta}^0(\xi)$ is not a tempered distribution, and the inverse Fourier transform is an analytical function in $\mathcal{Z}'$. Nevertheless, for $\varepsilon > 0$, $\hat{\beta}^\varepsilon(\xi)$ is "well-behaved" for $\xi \rightarrow +\infty$ as

$$\hat{\beta}^\varepsilon(\xi) \approx \frac{\hat{\varphi}(\xi)}{\xi \varepsilon^2}. \tag{3.23}$$

This specific behavior depends on that of $\frac{\hat{\varphi}}{\xi}$, so that in most cases will be decreasing, but multiplied by the factor ε^{-2}. When $\varepsilon > 0$ (small but not 0) is fixed, $\hat{\beta}^\varepsilon(\xi)$ is approximatively given by (3.21) for "finite" ξ and by (3.23) for ξ going to $+\infty$. It is easily seen that the sup in modulus of $|\hat{\beta}^\varepsilon(\xi)|$ is located in the region where both terms in the denominator of the right hand side of (3.20) are of the same order (so that no one of them may ble neglected). This gives

$$\xi = \mathcal{O}(log(1/\varepsilon)). \tag{3.24}$$

It appears that $\hat{\beta}^\varepsilon(\xi)$ consists mainly of Fourier components which tend to infinity algebraically as ε goes to zero with ξ tending to infinite "slowly" as in (3.24). this is somewhat analogous to the example, given in the introduction, of a sequence of functions converging to an analytical functional.

Coming back to (3.18), the main properties of the behavior of $u^\varepsilon(x, 1)$ may be thrown:

- The trace $u^\varepsilon(x, 1) = \beta^\varepsilon(x)$ on the boundary Γ_1 which bears the "pathological boundary condition" mainly consists of large oscillations with wave length $1/\log(1/\varepsilon)$ (which tends to 0 very slowly as $\varepsilon \to 0$). The amplitude of that oscillations grows nearly as ε^{-2}. The limit $\varepsilon \to 0$ does not exist in distribution theory; it constitutes a complexification process.

- Out of the trace on Γ_1, (i. e. for $0 < y < 1$), the behavior is analogous, but of lower amplitude, which is exponentially decreasing going away of Γ_1. We recover properties of the non-uniqueness associated with the failed SL condition.

Before concluding this section, we would like to show some analogy between the previous limit problem and the Cauchy elliptic problem, which is a classical example of ill-posed problem, without solution in general.

We consider the same domain Ω as before, but we now impose two boundary conditions on Γ_0 and no condition on Γ_1. Namely

$$\begin{cases} \triangle v = 0 \text{ on } \Omega \\ v = \psi \text{ on } \Gamma_0 \\ \partial_y v = 0 \text{ on } \Gamma_0 \end{cases} \tag{3.25}$$

Taking as above the $x \to \xi$ Fourier transform, it follows immediately that

$$\hat{v}(\xi, y) = \hat{\psi}(\xi) \cosh(\xi y). \tag{3.26}$$

Where it is apparent that the behavior for $\xi \to \infty$ is exponentially growing (unless in the case when $\hat{\psi}(\xi)$ decays faster than $e^{-|\xi|}$) so that it is not tempered and the inverse Fourier transform does not exist within distribution theory.

4 A model variational sensitive singular perturbation, [EgMeSa07]

4.1 Setting of the problem

Let Ω be a two dimensional compact manifold with smooth (of C^∞ class) boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ of the variable $x = (x_1, x_2)$, where Γ_0 and Γ_1 are disjoint; they are one - dimensional compact smooth manifolds without boundary, then diffeomorphic to the unit circle. Let a and b be the bilinear forms given by:

$$a(u, v) = \int_\Omega \triangle u \, \triangle v \, \mathrm{d}x, \tag{4.27}$$

$$b(u, v) = \int_\Omega \sum_{\alpha,\beta=1}^{2} \partial_{\alpha\beta} u \, \partial_{\alpha\beta} v \, \mathrm{d}x. \tag{4.28}$$

We consider the following variational problem (which has possibly only a formal sense)

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{cases} \tag{4.29}$$

where the space V is the "energy space" with the essential boundary conditions on Γ_0

$$V = \{v \in H^2(\Omega);\ v_{|\Gamma_0} = \frac{\partial v}{\partial n}_{|\Gamma_0} = 0\}, \tag{4.30}$$

where n, t denotes the normal and tangent unit vectors to the boundary Γ with the convention that the normal vector n is inwards Ω. It is easily checked that the bilinear form b is coercive on V. Moreover, we immediately obtain the following result. For all $\varepsilon > 0$ and for all f in V', the variational problem (4.29) is of Lax-Milgram type and it is a self-adjoint problem which has a coerciveness constant larger than $c\varepsilon^2$, with $c > 0$.

The equation on Ω associated with problem (4.29) is:

$$(1 + \varepsilon^2)\triangle^2 u^\varepsilon = f \text{ on } \Omega, \tag{4.31}$$

as both forms a and b give the laplacian. As for the boundary conditions on Γ_0, they are "principal" i. e. they are included in the definition of V, (5.4). As for conditions on Γ_1, they are "natural", classically obtained from the integrated terms by parts. Those coming from the form b are somewhat complicated; we shall not write them, as the problem with $\varepsilon > 0$ is classical. For $\varepsilon = 0$ these conditions (coming from form a) are: $\triangle u = \frac{\partial \triangle u}{\partial n} = 0,$ on Γ_1.

As a matter of fact, the full limit boundary boundary value problem is:

$$\begin{cases} \triangle^2 u^0 = f \text{ on } \Omega \\ u = \frac{\partial u^0}{\partial n} = 0, \text{ on } \Gamma_0 \\ \triangle u^0 = 0 \text{ on } \Gamma_1 \\ -\frac{\partial}{\partial n}\triangle u^0 = 0 \text{ on } \Gamma_1. \end{cases} \tag{4.32}$$

Let us check that the boundary conditions on Γ_1 (i. e; the two last lines of (5.6)) do not satisfy the SL condition for the elliptic operator $\triangle^2$. Indeed, proceeding as in sect. 2, by formal tangential Fourier transform

$$(-\xi_1^2 + \partial_2^2)^2 \hat{u} = 0. \tag{4.33}$$

which yields that

$$\hat{v} = (Ae^{-|\xi_1|x_2} + Cx_2 e^{-|\xi_1|x_2}) \tag{4.34}$$

(as well as analogous terms with $+|\xi|$ instead of $-|\xi|$, which are not taken into account as exponentially growing inwards the domain). Here, according to SL theory, x_2 is the coordinate normal to the boundary, after taking locally tangent and normal axes, (which do not modify the equation $\triangle^2$). The (tangential Fourier transform of the) boundary conditions on Γ_1 are:

$$(-\xi_1^2 + \partial_2^2)\hat{u} = 0 \tag{4.35}$$

and

$$\partial_2(-\xi_1^2 + \partial_2^2)\hat{u} = 0. \tag{4.36}$$

It is immediately seen that the previous solutions (4.34) with $C = 0$ and any $A \neq 0$ satisfy both conditions (note that its laplacian vanishes everywhere, then it vanishes as well as its normal derivative on the boundary). So, the SL condition is not satisfied on Γ_1.

Before going on with our study, we note that the limit problem (4.32) implies an elliptic Cauchy problem for the auxiliary unknown

$$v^0 = \triangle u^0. \tag{4.37}$$

Indeed, system (4.32) gives in particular:

$$\begin{cases} \triangle v^0 = f \text{ on } \Omega \\ v^0 = 0 \text{ on } \Gamma_1 \\ -\frac{\partial v^0}{\partial n} = 0 \text{ on } \Gamma_1. \end{cases} \tag{4.38}$$

which is precisely the Cauchy problem for the laplacian.

As mentioned in section 3, this is a classical ill - posed problem, and the solution does not exist in general. Oppositely, uniqueness of the solution holds true (uniqueness theorem of Holmgren and analogous, see for instance [CoHi62]).

4.2 The heuristic integral approach

The aim of this section is the construction, in a heuristic way, of an approximate description of the solutions u^ε of the model problem in the previous section for small values of ε.

From the general theory of singular perturbations of the form (4.29), we know that our assumption

$$a(v,v)^{1/2} \text{ defines a norm on } V, \tag{4.39}$$

is crucial. Indeed, when it is not satisfied, the problem is said to be "non inhibited". In such a case, it has a kernel which contains non vanishing terms and then, it is easy to establish that the asymptotic behaviour of the solution u^ε of (4.29) is described by a variational problem in this kernel. The previous fact is not surprising as soon as we consider the following minimization problem, which is equivalent to (4.29),

$$\begin{cases} \text{Minimize in } V, \\ a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) - 2\langle f, u^\varepsilon \rangle. \end{cases} \tag{4.40}$$

Indeed, when ε goes to zero, the natural trend consists in avoiding the a-energy which occurs with the factor 1 and leaving the b-energy which has a factor ε^2.

Clearly, this is not possible when (4.39) is satisfied since the kernel reduces to the zero function. Nevertheless, in our case, $a(v,v) = 0$ implies $\triangle v = 0$ and, as $v \in V$, the traces of v and $\frac{\partial v}{\partial n}$ vanish on Γ_0, so that (4.39) follows from the uniqueness theorem for the Cauchy problem. This uniqueness is classical, but the solution u is unstable in the sense that there can be "large u" in the V norm (or in other spaces) for "small f" in the V' norm (or in other spaces). It then appears that the same reasoning shows that for small values of ε, the solution u^ε will be precisely among elements with small $a(u^\varepsilon, u^\varepsilon)$, that is to say with small $\triangle u^\varepsilon$ in L^2.

4.3 The Γ_0 layer

Let us now build such functions $u^\varepsilon \in V$ with very small$\|\triangle u^\varepsilon\|_{L^2}$. The main idea is to consider functions in a larger space than the space of functions v of V such that $\triangle v = 0$ (which only contains the function $v = 0$). The functions of this bigger space will not satisfy the two boundary conditions on Γ_0 that are satisfied by any function of V. Then we shall modify it in a narrow boundary layer along Γ_0 in order to satisfy the two boundary conditions with small value of a-energy.

More precisely, let us consider the vector space:

$$G^0 = \{v \in C^\infty(\overline{\Omega}),\ \triangle v = 0 \text{ on } \Omega,\ v = 0 \text{ on } \Gamma_0\}. \tag{4.41}$$

Remark 4.1. *We observe that every function of G^0 satisfies one of the boundary conditions on Γ_0 which are satisfied by any element of V. For simplicity, we have chosen $v = 0$ on Γ_0, but we could choose the other one $\frac{\partial v}{\partial n} = 0$ on Γ_0 as well. On the other hand, the regularity assumption C^∞ is slightly arbitrary. Since, we will consider the completion of G^0 with respect to some norm, this point is irrelevant.*

Obviously, as the Dirichlet problem for the laplacian on Ω is well posed in C^∞, the space G^0 is isomorphic with the space of traces on Γ_1:

$$\{w \in C^\infty(\Gamma_1)\} \tag{4.42}$$

the isomorphism is obtained by solving the Dirichlet problem:

$$\begin{cases} \triangle \tilde{w} = 0 \text{ on } \Omega, \\ \tilde{w} = 0 \text{ on } \Gamma_0, \\ \tilde{w} = w \text{ on } \Gamma_1. \end{cases} \tag{4.43}$$

In the sequel, we shall consider indifferently the functions $\tilde{w}$ on $\overline{\Omega}$ or their traces w on Γ_1.

In fact, the exact function u^ε is a solution of (4.31), which we are searching to describe approximatively in order to define a space as small as possible (incorporating the main features of the solution) to solve the minimization problem. More precisely, according to our previous comments, we are interested in the ”most singular parts” of u^ε in the sense of the part corresponding to the high frequency Fourier components. As we shall see in the sequel, it turns out that these singular parts may be obtained by modification of the functions $\tilde{w}$ on a boundary layer close to Γ_0; this layer is narrower when the considered Fourier components are of higher frequency; in fact, the layer only exists because we only consider high frequencies. This allows to make an approximation which consists in using locally curvilinear coordinates defined by the arc of Γ_0 and the normal, and handling them as cartesian coordinates. Clearly, this approximation is exact only on the very Γ_0, but more and more precise as we approach of Γ_0, i. e. as the considered frequencies grow.

Once the layer is constructed, we compute the a-energy of it, as well as the $\varepsilon^2 b$-energy of the (modified) $\tilde{w}$ function, in order to consider the variational problem (4.29) in the restricted space.

Let us first exhibit the local structure of the Fourier transform of $\tilde{w}$ close to Γ_0. According to our general considerations on the heuristic procedure, $\hat{w}$ may be considered (after multiplying by an appropriate cutoff function) of ”small support” near a point P_0 of Γ_0. Taking local tangent and normal cartesian coordinates y_1, y_2, we have, within our approximation,

$$\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\Big)\tilde{w} = 0 \text{ on } \mathbf{R} \times (0, t), \tag{4.44}$$

for some $t > 0$. Taking the tangential Fourier transform, we obtain:

$$\mathcal{F}(\tilde{w}_j)(\xi_1, y_2) = \lambda e^{|\xi_1| y_2} + \mu e^{-|\xi_1| y_2}. \tag{4.45}$$

It is worthwhile defining the local structure of $\hat{w}$ in the vicinity of Γ_0 using the ”Cauchy” data $\tilde{w}$ and $\partial_2 \tilde{w}$ on Γ_0 (note that the solution of the Cauchy problem is unique, so that the Cauchy data determine the solution). As $\hat{w}$ vanishes on Γ_0, the local structure is then determined by $\partial_2\tilde{w}$ on Γ_0. Taking the tangential Fourier transform this gives:

$$\mathcal{F}\Big(\tilde{w}_j\Big)(\xi_1, y_2) = \mathcal{F}\Big(\frac{\partial \tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\frac{\sinh(|\xi_1|y_2)}{|\xi_1|}. \tag{4.46}$$

We now proceed to the modification of $\tilde{w}$ into $\tilde{w}^a$ in a narrow boundary layer of Γ_0 in order to satisfy (always within our approximation) the equation coming from (4.31) for small ε. Using considerations similar to those leading to (4.44), this amounts to

$$\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\Big)^{(2)}\tilde{w}^a = 0 \text{ on } \mathbf{R}\times(0,t). \tag{4.47}$$

hence the tangential Fourier transform reads

$$\Big(-|\xi_1|^2 + \frac{\partial^2}{\partial y_2^2}\Big)^{(2)}\mathcal{F}(\tilde{w}^a) = 0. \tag{4.48}$$

Consequently, $\mathcal{F}(\tilde{w}^a)$ should take the form

$$\mathcal{F}(\tilde{w}^a)(\xi_1, y_2) = (\alpha + \gamma y_2)e^{|\xi_1|y_2} + (\beta + \delta y_2)e^{-|\xi_1|y_2}. \tag{4.49}$$

The four unknown constants should be determined by imposing that $\tilde{w}^a$ and $\partial_2\tilde{w}^a$ vanish for $y_2 = 0$ and the ”matching condition” of the layer, i.e., out of the layer, we want $\tilde{w}_j^a$ to match with the given function $\tilde{w}_j$. Since $|\xi_1| >> 1$, then $|\xi_1|y_2 >> 1$ means that $y_2 >> \frac{1}{|\xi_1|}$ (but we still impose that y_2 is small in order to be in a narrow layer of Γ_0); this is perfectly consistent, as we will only use the functions for large $|\xi_1|$, hence the terms with coefficients β and δ are ”boundary layer terms” going to zero out of the layer (i.e. for $|y_2| >> \mathcal{O}\Big(\frac{1}{|\xi_1|}\Big)$), see perhaps [Ec79] or [Il91] for generalities on boundary layers and matching. This gives

$$\mathcal{F}\Big(\tilde{w}_j\Big)(\xi_1, y_2) = \mathcal{F}\Big(\frac{\partial \tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\Big(\frac{\sinh(|\xi_1|y_2)}{|\xi_1|} - y_2 e^{-|\xi_1|y_2}\Big). \tag{4.50}$$

This amounts to saying that the modification of the function $\tilde{w}_j$ consists in adding to it the inverse Fourier transform of

$$\mathcal{F}\Big(\frac{\partial \tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\Big(-y_2 e^{-|\xi_1|y_2}\Big). \tag{4.51}$$

Defining on Γ_0 the family (with parameter y_2) of pseudo-differential smoothing operators $\delta\sigma(\varepsilon, D_1, y_2)$ with symbol:

$$\delta\sigma(\varepsilon, \xi_1, y_2) = -y_2 e^{-|\xi_1|y_2}h(\varepsilon, \xi, y_2), \tag{4.52}$$

where h is an irrelevant cutoff function avoiding low frequencies; it is equal to 1 for high frequencies (see [EgMeSa07] for details), we see that the modification of the function $\tilde{w}$:

$$\delta\tilde{w} = \tilde{w}^a - \tilde{w} \tag{4.53}$$

is precisely the action of $\delta\sigma(\varepsilon, D_1, y_2)$ on $\frac{\partial \tilde{w}_j}{\partial y_2}(y_1, 0)$:

$$\delta\tilde{w} = \delta\sigma(\varepsilon, D_1, y_2)\frac{\partial \tilde{w}_j}{\partial y_2}(y_1, 0). \tag{4.54}$$

Let us now compute the leading terms of the a-energy of the modified function $\tilde{w}^a$.

Let $\tilde{v}$ and $\tilde{w}$ be two elements in G^0 and $\tilde{v}^a, \tilde{w}^a$ the corresponding elements modified in the boundary layer. As the given $\tilde{v}$ and $\tilde{w}$ are harmonic in Ω, the a-form is only concerned with the modification terms $\delta\tilde{v}$ and $\delta\tilde{w}$. Then, within our approximation, we have:

$$a(\tilde{v}^a, \tilde{w}^a) = \int_{\Gamma_0} dy_1 \int_0^{+\infty} \triangle(\delta\tilde{v})\overline{\triangle(\delta\tilde{w})}\, \mathrm{d}y_2. \tag{4.55}$$

To compute this expression, we first write $\tilde{v}$ and $\tilde{w}$ as sum of terms with "small support" (by multiplying by a partition of unity): $\tilde{v} = \Sigma_j \tilde{v}_j$ and $\tilde{w} = \Sigma_j \tilde{w}_j$. Then, within our approximation, the integral is on the halfplane $\mathbf{R} \times (0, +\infty)$ of the variables y_1, y_2. Taking the tangential Fourier transform and using the Parceval-Plancherel theorem, we have

$$a(\tilde{v}^a, \tilde{w}^a) = \Sigma_{j,k} \int_{-\infty}^{+\infty} \mathrm{d}\xi_1 \int_0^{+\infty} \Big(\frac{\mathrm{d}^2}{\mathrm{d}y_2^2} - \xi_1^2\Big)\delta\sigma(\varepsilon, \xi, y_2)\mathcal{F}\big(\frac{\partial\tilde{v}_j}{\partial y_2}_{|y_2=0}\big) \times \overline{\Big(\frac{\mathrm{d}^2}{\mathrm{d}y_2^2} - \xi_1^2\Big)\delta\sigma(\varepsilon, \xi, y_2)\mathcal{F}\big(\frac{\partial\tilde{w}_k}{\partial y_2}_{|y_2=0}\big)}\, \mathrm{d}y_2.$$

Hence, on account of (4.52) and integrating in y_2, this yields

$$a(\tilde{v}^a, \tilde{w}^a) = \Sigma_{j,k} \int_{-\infty}^{+\infty} 2|\xi_1| \frac{\partial\tilde{w}_{1,j}}{\partial y_2}_{|y_2=0} \overline{\frac{\partial\tilde{w}_{2,k}}{\partial y_2}_{|y_2=0}} h^2(\varepsilon, \xi, y_2)\, \mathrm{d}\xi_1. \tag{4.56}$$

This expression (4.56) only depends on the traces $\frac{\partial\tilde{v}_j}{\partial y_2}_{|y_2=0}(y_1)$ and $\frac{\partial\tilde{w}_k}{\partial y_2}_{|y_2=0}(y_1)$, which are functions defined on Γ_0.

We now simplify this last expression using a sesquilinear form involving pseudo-differential operators.

Indeed, denoting by $P(\frac{\partial}{\partial y_1})$ the pseudo-differential operator with symbol

$$P(\xi_1) = (2|\xi_1|)^{1/2} h(\varepsilon, \xi, y_2), \tag{4.57}$$

and summing over j and k, we obtain that

$$a(\tilde{v}^a, \tilde{w}^a) = \int_{\Gamma_0} P(\frac{\partial}{\partial s})\frac{\partial\tilde{v}}{\partial n}_{|\Gamma_0} \overline{P(\frac{\partial}{\partial s})\frac{\partial\tilde{w}}{\partial n}_{|\Gamma_0}}\, \mathrm{d}s. \tag{4.58}$$

4.4 Taking account of the perturbation term $\varepsilon^2 b$.

We now consider the minimization problem (4.40) on G^0 instead of on V. Obviously, the a-energy should be computed using formula (4.58). This modified problem should involve the a-energy and the $\varepsilon^2 b$-energy. A natural space for handling it should be the completion G of G^0 with the norm:

$$\|v\|_G^2 = \int_{\Gamma_0} \Big| P(\frac{\partial}{\partial s})\frac{\partial v}{\partial n}_{|\Gamma_0}\Big|^2 \mathrm{d}s + b(v, v). \tag{4.59}$$

It is easily seen that G is the space of the harmonic functions of $H^2(\Omega)$ vanishing on Γ_0; according to (4.43) it may be identified with the space of traces $H^{3/2}(\Gamma_1)$.

It will prove useful to write another (asymptotically equivalent for large $|\xi_1|$) definition of this problem. Indeed, the elements $\tilde{w}$ of G^0 (and then of G) may be identified (by solving the problem

(4.43)) with their traces w on Γ_1. Moreover, as the functions $\tilde{w}$ are harmonic, we may exhibit their local behavior in the vicinity of any point $x_0 \in \Gamma_1$. Proceeding as in (4.44), (4.45) and taking only the decreasing exponential towards the domain (this is the classical approximation for the construction of a parametrix) we have:

$$\mathcal{F}(\tilde{w})(\xi_1, y_2) = \mathcal{F}(w)(\xi_1)e^{-|\xi_1|y_2}, \tag{4.60}$$

where y_1, y_2 are the tangent and the normal (inwards the domain) vectors. Then, it is apparent that the b-energy is concentrated in a layer close to Γ_1 and we may compute it in an analogous way to the calculus that was done for the a-energy (4.58). Indeed, using Parseval-Plancherel Theorem and within our approximation, we have

$$\begin{aligned} b(\tilde{w}, \tilde{w}) &= \int_{-\infty}^{+\infty} \mathrm{d}y_1 \int_0^{+\infty} \sum_{\alpha,\beta} |\partial_{\alpha\beta}\tilde{w}|^2 \, \mathrm{d}y_2 \qquad (4.61) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\xi_1 \int_0^{+\infty} \Big(\xi_1^4|\mathcal{F}(\tilde{w})|^2 + 2\xi_1^2|\mathcal{F}(\frac{\partial \tilde{w}}{\partial y_2})|^2 + |\mathcal{F}(\frac{\partial^2 \tilde{w}}{\partial y_2^2})|^2\Big) \, \mathrm{d}y_2, \end{aligned}$$

hence, recalling (4.60) and integrating over y_2, we get:

$$b(\tilde{w}, \tilde{w}) = 2\int_{-\infty}^{+\infty} |\xi_1|^3|\mathcal{F}(w)|^2 \, \mathrm{d}\xi_1. \tag{4.62}$$

Then, defining the pseudo-differential operator $Q(\frac{\partial}{\partial s})$ of order $3/2$ with principal symbol

$$\sqrt{2}|\xi_1|^{3/2}, \tag{4.63}$$

or equivalently as previously:

$$\sqrt{2}(1 + |\xi_1|^2)^{3/4}, \tag{4.64}$$

we have (always within our approximation):

$$b(\tilde{v}, \tilde{w}) = \int_{\Gamma_1} Q(\frac{\partial}{\partial s})v \, \overline{Q(\frac{\partial}{\partial s})w} \, \mathrm{d}s. \tag{4.65}$$

We observe that the operator Q is only concerned with the trace on Γ_1, so that we may either write $\tilde{v}$, $\tilde{w}$ or v, w in (4.65).

The formal asymptotic problem becomes:

$$\left\{ \begin{array}{l} \text{Find } \tilde{v}^\varepsilon \in G \text{ such that } \forall \tilde{w} \in G \\ \int_{\Gamma_0} P(\frac{\partial \tilde{v}^\varepsilon}{\partial n})\overline{P(\frac{\partial \tilde{w}}{\partial n})} \, \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\tilde{v}^\varepsilon) \, \overline{Q(\tilde{w})} \, \mathrm{d}s = \langle f, w \rangle. \end{array} \right. \tag{4.66}$$

4.5 The formal asymptotics and its sensitive behaviour

In order to exhibit more clearly the unusual character of the problem, we shall now write (4.66) under another equivalent form involving only the traces on Γ_1. Coming back to (4.43), let us define $\mathcal{R}_0$ as follows. For a given $w \in C^\infty(\Gamma_1)$ we solve (4.43) and we take the trace of $\frac{\partial \tilde{w}}{\partial n}$ on Γ_0, then

$$\frac{\partial \tilde{w}}{\partial n}_{|\Gamma_0} = \mathcal{R}_0 w. \tag{4.67}$$

Using the regularity properties of the solution of (4.43), it follows that $\mathcal{R}_0 w$ is in $C^\infty(\Gamma_0)$. In fact, R_0 is a smoothing operator, sending any distribution into a C^∞ function. Then, (4.66) may be written as a problem for the traces on Γ_1:

$$\left\{ \begin{array}{l} \text{Find } v^\varepsilon \in H^{3/2}(\Gamma_1) \text{ such that } \forall w \in H^{3/2}(\Gamma_1) \\ \int_{\Gamma_0} P(\frac{\partial}{\partial s})\mathcal{R}_0 v^\varepsilon \overline{P(\frac{\partial}{\partial s})\mathcal{R}_0 w}\, \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\frac{\partial}{\partial s}) v^\varepsilon\, \overline{Q(\frac{\partial}{\partial s}) w}\, \mathrm{d}s = \int_\Omega F \tilde{w}\, \mathrm{d}x, \end{array} \right. \tag{4.68}$$

where the configuration space is obviously $H^{3/2}(\Gamma_1)$. The left hand side with $\varepsilon > 0$ is continuous and coercive. We then define the new operators

$$\mathcal{A} = \mathcal{R}_0^* P^* P \mathcal{R}_0 \in \mathcal{L}(H^s(\Gamma_1), H^r(\Gamma_0)), \forall s, r \in \mathbf{R}, \tag{4.69}$$

$$\mathcal{B} = Q^* Q \in \mathcal{L}(H^{3/2}(\Gamma_1), H^{-3/2}(\Gamma_1) \tag{4.70}$$

where $\mathcal{R}_0^*$ is the adjoint of $\mathcal{R}_0$ (which is also smoothing)), (4.68) becomes

$$\Big(\mathcal{A} + \varepsilon^2 \mathcal{B}\Big) v^\varepsilon = F, \text{ in } H^{-3/2}(\Gamma_1). \tag{4.71}$$

Obviously, $\mathcal{B}$ is an elliptic pseudo-differential operator of order 3, whereas $\mathcal{A}$ is a smoothing (non local) operator.

This problem is somewhat simpler than the initial one (as on a manifold of dimension 1), showing the interest of the formal asymptotics. It enters in a class of sensitive problems addressed in [EgMeSa07] section 2. It is apparent that the limit problem (for $\varepsilon = 0$) has no solution in the distribution space for any F not contained in $\mathcal{C}^\infty$. Indeed, on the compact manifold Γ_0, any distribution is in some $H^{-m}(\Gamma_0)$ space, which is send into $\mathcal{C}^\infty$ by the smoothing operator $\mathcal{A}$.

Remark 4.2. *The drastically non local character of the smoothing operator $\mathcal{A}$ follows from the fact that it involves $\mathcal{R}_0$ and $\mathcal{R}_0^*$ (see(4.67)). This is the reason why the problem may be reduced to another one on the traces on Γ_1. The possibility of that reduction is a consequence of our approximation, where the configuration space is formed by harmonic functions.*

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